BOUNDARY BEHAVIOR FOR A SINGULAR QUASI-LINEAR ELLIPTIC EQUATION

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Abstract. In a smooth bounded domain we obtain existence and uniqueness, regularity and boundary behavior for a class of singular quasi-linear elliptic equations.

1. Introduction and result

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. The aim of this note is to establish existence, uniqueness and boundary behavior of the solutions to the singular quasi-linear problem

$$\begin{cases} -\mathrm{div}(a(u)Du) + \frac{a'(u)}{2}|Du|^2 = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a : (0, +\infty) \to (0, +\infty)$ is a C^1 function bounded away from zero, $f : (0, +\infty) \to (0, +\infty)$ is a C^1 function and there exist $a_0 > 0$ and $f_0 > 0$ such that

$$\lim_{s\to 0^+} a(s)s^{2\mu} = a_0, \qquad \text{for some } 0 \le \mu < 1, \tag{1.2}$$

$$\lim_{s\to 0^+} f(s)s^{\gamma} = f_0, \qquad \text{for some } \gamma > 1, \tag{1.3}$$

$$2f'(s)a(s) \le f(s)a'(s), \qquad \text{for every } s > 0. \tag{1.4}$$

We refer to [11] and to the references included for the interest and motivations to analyze these equations. In the one dimensional case, equations such as (1.1) typically arise in certain problems in fluid mechanics and pseudo-plastic flow (see e.g. [9,12]). In the semi-linear case $a \equiv 1$ various results about existence, uniqueness and asymptotic behavior of the solutions have been obtained in the literature so far (see [2,4,7,8,13], the monographs [3,10] and the references therein). Under assumptions (1.2)-(1.4), if $\mathrm{d}(x, \partial\Omega)$ denotes the distance of a point x in Ω from the boundary $\partial\Omega$, we shall prove the following

Theorem 1.1. *Problem* (1.1) *has a unique solution* $u \in C^2(\Omega) \cap C(\bar{\Omega})$ *and there exist* $\Gamma, \Gamma', \Gamma'' > 0$ *with*

$$\Gamma \mathrm{d}(x, \partial\Omega)^{\frac{2}{1+\gamma-2\mu}} \le u(x) \le \Gamma' \mathrm{d}(x, \partial\Omega)^{\frac{2}{1+\gamma-2\mu}}, \quad |Du(x)| \le \Gamma'' \mathrm{d}(x, \partial\Omega)^{\frac{1-\gamma+2\mu}{1+\gamma-2\mu}}, \quad \text{as } \mathrm{d}(x, \partial\Omega) \to 0.$$

Moreover $u \in \mathrm{Lip}(\bar{\Omega})$ *if* $1 < \gamma \le 1 + 2\mu$, $u \in C^{0,\frac{2}{1+\gamma-2\mu}}(\bar{\Omega})$ *if* $\gamma > 1 + 2\mu$ *and* $u \in H^1_0(\Omega)$ *if* $\gamma < 3 - 2\mu$.

Whence, in some sense, the functions a and f compete for the vanishing rate of the solution and for its gradient upper bound, near the boundary $\partial\Omega$. Furthermore, the range for Lipschitz continuity of u up to the boundary is $\gamma \le 1 + 2\mu$, $\gamma \neq 1$ thus enlarged with respect to the one for the semi-linear case, namely $0 < \gamma < 1$, see also Remark 2.2. In [5], the author jointly with F. Gladiali have recently performed a complete study about existence and qualitative behavior around $\partial\Omega$ of the solutions to the problem

$$\begin{cases} \mathrm{div}(a(u)Du) - \frac{a'(u)}{2}|Du|^2 = f(u) & \text{in } \Omega, \\ u(x) \to +\infty & \text{as } \mathrm{d}(x, \partial\Omega) \to 0, \end{cases}$$

covering situations where a and f have an exponential, polynomial or logarithmic type growth at infinity and a nonsingular behavior around the origin. On the contrary, here we focus on the singular behavior at the origin for a and f with the action of the source f being in some sense predominant at zero upon the diffusion a, due to the constraint $\gamma > 1 > \mu$. Without loss of generality we assume that a grows as s^k and f

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decays as s^{-p} as $s \to +\infty$ for some $k \geq 0$ and $p > 1$, in which case one can also obtain estimates for u and $|Du|$ valid on the whole $\bar{\Omega}$, as pointed out in Remark 2.2. In the particular case when $a \equiv 1$, problem (1.1) reduces to $-\Delta u = f(u)$ and the above estimates reduce to $\Gamma \mathrm{d}(x, \partial\Omega)^{2/(1+\gamma)} \leq u(x) \leq \Gamma' \mathrm{d}(x, \partial\Omega)^{2/(1+\gamma)}$ and $|Du(x)| \leq \Gamma'' \mathrm{d}(x, \partial\Omega)^{(1-\gamma)/(1+\gamma)}$ as $\mathrm{d}(x, \partial\Omega) \to 0$, consistently with the results of [2,13]. Following the line of [7,13], some easy adaptations of Theorem 1.1 can be obtained to cover the case of non-autonomous nonlinearities such as $q(x)f(u)$ in place of $f(u)$ and of unbounded domains of $\mathbb{R}^N$. We leave these further developments to the interested reader. As an example of f and a satisfying (1.2)-(1.4) one can take $f(s) = s^{-\gamma}$, $a(s) = s^{-2\mu}$ for $s \leq s_0$ and $a(s) = \theta(s)$ for $s \geq s_0$ for some $s_0 > 0$, with $\theta \in C^1 \cap L^\infty$ bounded away from zero, $\theta(s_0) = s_0^{-2\mu}$, $\theta'(s_0) = -2\mu s_0^{-2\mu-1}$ and $s\theta'(s) + 2\gamma\theta(s) \geq 0$ for $s \geq s_0$.

2. Proof of the result

In this section, we prove Theorem 1.1. We shall assume that conditions (1.2)-(1.4) hold. In order to get information about existence, uniqueness and the boundary behavior of the solutions to (1.1), we convert the quasi-linear problem (1.1) into a corresponding semi-linear problem through a change of variable procedure involving the Cauchy problem for $g \in C^2((0, +\infty)) \cap C([0, +\infty))$,

$$
\begin{cases} g'(s) = \frac{1}{\sqrt{a(g(s))}}, & \text{for } s > 0, \\ g(0) = 0, \\ g(s) > 0, & \text{for } s > 0. \end{cases} \tag{2.1}
$$

Due to the requirement $g > 0$, the solutions of (2.1) are unique and solve $\int_0^{g(s)} \sqrt{a(\xi)} d\xi = s$, for $s > 0$. The solution is global for $s > 0$ since a is bounded away from zero. This procedure was also followed in [5] in the framework of explosive solutions, although there g is C^2 around the origin and defined on $\mathbb{R}$. Now, since $g \in C^2((0, +\infty)) \cap C([0, +\infty))$ and it is strictly increasing, it is readily seen by a direct computation that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a positive solution to (1.1) if and only if $v = g^{-1}(u) \in C^2(\Omega) \cap C(\bar{\Omega})$ is a positive solution to $-\Delta v = h(v)$ in Ω, where we have set $h(s) := f(g(s))/\sqrt{a(g(s))}$ for $s > 0$. Let us now obtain the asymptotic behavior of the solution g to problem (2.1) as $s \to 0^+$ depending of the assigned asymptotic behavior of a as $s \to 0^+$, given by (1.2). For every $0 \leq \mu < 1$, we have

$$
\lim_{s \to 0^+} \frac{g(s)}{s^{\frac{1}{1-\mu}}} = (1-\mu)^{\frac{1}{1-\mu}} a_0^{\frac{1}{2(\mu-1)}}. \tag{2.2}
$$

In fact, taking into account (2.1), by l'Hôpital's rule we have

$$
\begin{aligned}
\lim_{s \to 0^+} \frac{g(s)}{s^{\frac{1}{1-\mu}}} &= (1-\mu) \lim_{s \to 0^+} \frac{g'(s)}{s^{\frac{\mu}{1-\mu}}} = (1-\mu) \lim_{s \to 0^+} \frac{1}{s^{\frac{\mu}{1-\mu}} \sqrt{a(g(s))}} \\
&= (1-\mu) \lim_{s \to 0^+} \frac{g^\mu(s)}{s^{\frac{\mu}{1-\mu}} \sqrt{a(g(s)) g^{2\mu}(s)}} = \frac{(1-\mu)}{\sqrt{a_0}} \Big(\lim_{s \to 0^+} \frac{g(s)}{s^{\frac{1}{1-\mu}}} \Big)^\mu,
\end{aligned}
$$

which yields the claim. Moreover, by virtue of (1.2), (1.3) and (2.2), we have

$$
\lim_{s \to 0^+} \frac{h(s)}{s^{\frac{\mu-\gamma}{1-\mu}}} = \lim_{s \to 0^+} f(g(s)) g(s)^\gamma \frac{1}{\sqrt{a(g(s)) g(s)^{2\mu}}} \Big[\frac{g(s)}{s^{\frac{1}{1-\mu}}} \Big]^{\mu-\gamma} = f_0 a_0^{\frac{1-\gamma}{2(\mu-1)}} (1-\mu)^{-\frac{\gamma-\mu}{1-\mu}}. \tag{2.3}
$$

Observe also that, since $a(s) \sim a_\infty s^k$ as $s \to +\infty$ and $f(s) \sim f_\infty s^{-p}$ as $s \to +\infty$ for some $a_\infty, f_\infty > 0$, $k \geq 0$ and $p > 1$, if g still denotes the solution to (2.1), we have three facts:

$$
\int_1^{+\infty} \frac{f(g(s))}{\sqrt{a(g(s))}} ds < +\infty, \quad \lim_{s \to 0^+} \frac{f(g(s))}{\sqrt{a(g(s))}} = +\infty, \quad s \mapsto \frac{f(g(s))}{\sqrt{a(g(s))}} \text{ is nonincreasing.} \tag{2.4}
$$

The first property follows immediately from the limit

$$
\lim_{s \to +\infty} \frac{g(s)}{s^{\frac{2}{k+2}}} = \big(\frac{k+2}{2} \frac{1}{\sqrt{a}_\infty} \big)^{\frac{2}{k+2}},
$$

which was proved in [5]. The other properties follow by (2.3) and (1.4), respectively. By virtue of (2.2) we now prove that, for every $\gamma > 1$ and $0 \leq \mu < 1$, there holds

$$\lim_{s\to 0^+} \frac{\int_{g(s)}^{+\infty} f(\xi)d\xi}{s^{\frac{1-\gamma}{1-\mu}}} = f_0 a_0^{\frac{\gamma-1}{2(1-\mu)}} (\gamma-1)^{-1}(1-\mu)^{-\frac{\gamma-1}{1-\mu}}. \tag{2.5}$$

In fact, since $\int_{g(s)}^{+\infty} f(\xi)d\xi \to +\infty$ for all $\gamma > 1$, it follows

$$\begin{aligned}
\lim_{s\to 0^+} \frac{\int_{g(s)}^{+\infty} f(\xi)d\xi}{s^{\frac{1-\gamma}{1-\mu}}} &= \frac{\mu-1}{1-\gamma} \lim_{s\to 0^+} \frac{f(g(s))}{s^{\frac{\mu-\gamma}{1-\mu}}\sqrt{a(g(s))}} = \frac{\mu-1}{1-\gamma} \lim_{s\to 0^+} \frac{f(g(s))g^{\mu}(s)}{s^{\frac{\mu-\gamma}{1-\mu}}\sqrt{a(g(s))g^{2\mu}(s)}} \\
&= \frac{\mu-1}{(1-\gamma)\sqrt{a_0}} \lim_{s\to 0^+} f(g(s))g^{\gamma}(s) \lim_{s\to 0^+} \frac{g^{\mu-\gamma}(s)}{s^{\frac{\mu-\gamma}{1-\mu}}} \\
&= \frac{\mu-1}{(1-\gamma)} \frac{f_0}{\sqrt{a_0}} \lim_{s\to 0^+} \frac{g^{\mu-\gamma}(s)}{s^{\frac{\mu-\gamma}{1-\mu}}} = \frac{\mu-1}{(1-\gamma)} \frac{f_0}{\sqrt{a_0}} \Big(\lim_{s\to 0^+} \frac{g(s)}{s^{\frac{1}{1-\mu}}} \Big)^{\mu-\gamma} \\
&= f_0 a_0^{\frac{\gamma-1}{2(1-\mu)}} (\gamma-1)^{-1}(1-\mu)^{-\frac{\gamma-1}{1-\mu}}.
\end{aligned}$$

Now, for every $\ell \geq 0$, there exists a unique solution $\phi \in C([0,+\infty)) \cap C^2((0,+\infty))$ of the problem

$$\begin{cases} \phi'(s) = \sqrt{\ell^2 + 2\displaystyle\int_{g(\phi(s))}^{+\infty} f(\xi)d\xi}, & \text{for } s > 0, \\ \phi(0) = 0, \quad \phi(s) > 0, & \text{for } s > 0, \\ \displaystyle\lim_{s\to+\infty} \phi'(s) = \ell, \quad \lim_{s\to+\infty} \phi(s) = +\infty. \end{cases} \tag{2.6}$$

To prove this, taking into account (2.4), it is sufficient to apply [13, Lemma 1.3]. Notice that, in particular, the solutions to problem (2.6) locally (namely for every fixed $a > 0$) solve the second order problem

$$\begin{cases} -\phi''(s) = \dfrac{f(g(\phi(s))}{\sqrt{a(g(\phi(s)))}}, & \text{for } 0 < s \leq a, \\ \phi(0) = 0, \quad \phi(s) > 0, & \text{for } 0 < s \leq a. \end{cases} \tag{2.7}$$

We can now prove that, for every $\gamma > 1$, $0 \leq \mu < 1$ and $\ell \geq 0$, there holds

$$\lim_{s\to 0^+} \frac{\phi(s)}{s^{\frac{2-2\mu}{1+\gamma-2\mu}}} = \Big(\frac{1+\gamma-2\mu}{2-2\mu}\Big)^{\frac{2-2\mu}{1+\gamma-2\mu}} f_0^{\frac{1-\mu}{1+\gamma-2\mu}} a_0^{\frac{\gamma-1}{2(1+\gamma-2\mu)}} (\gamma-1)^{\frac{\mu-1}{1+\gamma-2\mu}} (1-\mu)^{\frac{1-\gamma}{1+\gamma-2\mu}}, \tag{2.8}$$

where ϕ denotes the unique solution to (2.6). In fact, by l'Hôpital's rule and (2.5), we obtain

$$\begin{aligned}
\lim_{s\to 0^+} \frac{\phi(s)}{s^{\frac{2-2\mu}{1+\gamma-2\mu}}} &= \lim_{s\to 0^+} \Big[\frac{\phi(s)^{\frac{1+\gamma-2\mu}{2-2\mu}}}{s}\Big]^{\frac{2-2\mu}{1+\gamma-2\mu}} = \lim_{s\to 0^+} \left[\frac{\phi(s)^{\frac{1+\gamma-2\mu}{2-2\mu}}}{\displaystyle\int_0^{\phi(s)} \frac{d\tau}{\sqrt{\ell^2 + \int_{g(\tau)}^{+\infty} f(\xi)d\xi}}}\right]^{\frac{2-2\mu}{1+\gamma-2\mu}} \\
&= \Big(\frac{1+\gamma-2\mu}{2-2\mu}\Big)^{\frac{2-2\mu}{1+\gamma-2\mu}} \lim_{s\to 0^+} \Big[\phi(s)^{\frac{\gamma-1}{2-2\mu}} \sqrt{\ell^2 + \int_{g(\phi(s))}^{+\infty} f(\xi)d\xi}\Big]^{\frac{2-2\mu}{1+\gamma-2\mu}} \\
&= \Big(\frac{1+\gamma-2\mu}{2-2\mu}\Big)^{\frac{2-2\mu}{1+\gamma-2\mu}} \lim_{s\to 0^+} \Big(\frac{\ell^2 + \int_{g(\phi(s))}^{+\infty} f(\xi)d\xi}{\phi(s)^{\frac{1-\gamma}{1-\mu}}}\Big)^{\frac{1-\mu}{1+\gamma-2\mu}} \\
&= \Big(\frac{1+\gamma-2\mu}{2-2\mu}\Big)^{\frac{2-2\mu}{1+\gamma-2\mu}} \lim_{s\to 0^+} \Big(\frac{\int_{g(s)}^{+\infty} f(\xi)d\xi}{s^{\frac{1-\gamma}{1-\mu}}}\Big)^{\frac{1-\mu}{1+\gamma-2\mu}} \\
&= \Big(\frac{1+\gamma-2\mu}{2-2\mu}\Big)^{\frac{2-2\mu}{1+\gamma-2\mu}} f_0^{\frac{1-\mu}{1+\gamma-2\mu}} a_0^{\frac{\gamma-1}{2(1+\gamma-2\mu)}} (\gamma-1)^{\frac{\mu-1}{1+\gamma-2\mu}} (1-\mu)^{\frac{1-\gamma}{1+\gamma-2\mu}}.
\end{aligned}$$

We are now ready to conclude the proof of Theorem 1.1. In light of [2, Theorem 1.1], since $h(s) \to +\infty$ as $s \to 0^+$ by (2.4) and h is non-increasing for $s > 0$ by (1.4), there exists a unique positive solution $z \in C^2(\Omega) \cap C(\bar{\Omega})$ to $-\Delta v = h(v)$. Then $g(z) \in C^2(\Omega) \cap C(\bar{\Omega})$ is a positive solution to (1.1). Assume that $u_1, u_2 \in C^2(\Omega) \cap C(\bar{\Omega})$, $u_1, u_2 > 0$ solve (1.1). Then $g^{-1}(u_1), g^{-1}(u_2) > 0$ solve $-\Delta v = h(v)$. By uniqueness, we deduce $g^{-1}(u_1) = g^{-1}(u_2)$, in turn yielding $u_1 = u_2$. By virtue of [2, Theorem 2.2 and Theorem 2.5] for any solution v of $-\Delta v = h(v)$ there exist four constants $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 > 0$ such that for $\mathrm{d}(x, \partial\Omega)$ small enough, $\Lambda_1 \phi(\mathrm{d}(x, \partial\Omega)) \leq v(x) \leq \Lambda_2 \phi(\mathrm{d}(x, \partial\Omega))$ and $|Dv(x)| \leq \Lambda_3 \big[\mathrm{d}(x, \partial\Omega) h(\Lambda_4 \phi(\mathrm{d}(x, \partial\Omega))) + \phi(\mathrm{d}(x, \partial\Omega))/\mathrm{d}(x, \partial\Omega)\big]$, being ϕ a solution to (2.7). On account of formula (2.8), we can find two constants $\Theta_1, \Theta_2 > 0$ such that, for $\mathrm{d}(x, \partial\Omega)$ small enough

$$\Theta_1 \mathrm{d}(x, \partial\Omega)^{\frac{2-2\mu}{1+\gamma-2\mu}} \leq \phi(\mathrm{d}(x, \partial\Omega)) \leq \Theta_2 \mathrm{d}(x, \partial\Omega)^{\frac{2-2\mu}{1+\gamma-2\mu}}, \tag{2.9}$$

which yield in turn

$$\Lambda_1 \Theta_1 \mathrm{d}(x, \partial\Omega)^{\frac{2-2\mu}{1+\gamma-2\mu}} \leq v(x) \leq \Lambda_2 \Theta_2 \mathrm{d}(x, \partial\Omega)^{\frac{2-2\mu}{1+\gamma-2\mu}}.$$

Finally, since g is increasing and $u = g(v)$, we have

$$g\big(\Lambda_1 \Theta_1 \mathrm{d}(x, \partial\Omega)^{\frac{2-2\mu}{1+\gamma-2\mu}}\big) \leq u(x) \leq g\big(\Lambda_2 \Theta_2 \mathrm{d}(x, \partial\Omega)^{\frac{2-2\mu}{1+\gamma-2\mu}}\big).$$

Finally, using (2.2), we obtain the desired controls on u. Now, from (2.9), as $\mathrm{d}(x, \partial\Omega)$ is small,

$$\phi(\mathrm{d}(x, \partial\Omega))/\mathrm{d}(x, \partial\Omega) \leq \Theta_2 \mathrm{d}(x, \partial\Omega)^{-\frac{\gamma-1}{1+\gamma-2\mu}}.$$

On account of (2.3) and (2.8), there exists a constant $\Theta_3 > 0$ such that

$$\begin{aligned}
&\mathrm{d}(x, \partial\Omega) h(\Lambda_4 \phi(\mathrm{d}(x, \partial\Omega))) = \\
&= \Lambda_4^{(\mu-\gamma)/(1-\mu)} \mathrm{d}(x, \partial\Omega) \Big[\frac{h(\Lambda_4 \phi(\mathrm{d}(x, \partial\Omega)))}{\Lambda_4^{(\mu-\gamma)/(1-\mu)} \phi^{(\mu-\gamma)/(1-\mu)}(\mathrm{d}(x, \partial\Omega))}\Big] \phi^{(\mu-\gamma)/(1-\mu)}(\mathrm{d}(x, \partial\Omega)) \\
&\leq \Theta_3 \mathrm{d}(x, \partial\Omega) \mathrm{d}(x, \partial\Omega)^{\frac{2\mu-2\gamma}{1+\gamma-2\mu}} = \Theta_3 \mathrm{d}(x, \partial\Omega)^{-\frac{\gamma-1}{1+\gamma-2\mu}},
\end{aligned}$$

for $\mathrm{d}(x, \partial\Omega)$ small enough, yielding $|Dv(x)| \leq \Lambda_3(\Theta_2 + \Theta_3) \mathrm{d}(x, \partial\Omega)^{-\frac{\gamma-1}{1+\gamma-2\mu}}$. Then, we have

$$\begin{aligned}
|Du(x)| = g'(v(x))|Dv(x)| &= \frac{|Dv(x)|}{\sqrt{a(g(v(x)))}} = \frac{|Dv(x)|}{\sqrt{a(g(v(x)))g^{2\mu}(v(x))}} g^{\mu}(v(x)) \\
&= \frac{1}{\sqrt{a(g(v(x)))g^{2\mu}(v(x))}} \frac{g^{\mu}(v(x))}{v(x)^{\mu/(1-\mu)}} v(x)^{\mu/(1-\mu)} |Dv(x)| \leq \omega_1 v(x)^{\mu/(1-\mu)} |Dv(x)| \\
&\leq \omega_2 \mathrm{d}(x, \partial\Omega)^{\frac{2\mu}{1+\gamma-2\mu}} \mathrm{d}(x, \partial\Omega)^{-\frac{\gamma-1}{1+\gamma-2\mu}} = \omega_2 \mathrm{d}(x, \partial\Omega)^{\frac{1-\gamma+2\mu}{1+\gamma-2\mu}}.
\end{aligned}$$

for some $\omega_1, \omega_2 > 0$. In particular, if $1 < \gamma \leq 1 + 2\mu$, it follows that u is Lipschitz continuous up the boundary. If instead $\gamma > 1 + 2\mu$, by the above estimates for u and $|Du|$, we find $\Theta_4 > 0$ such that

$$\begin{aligned}
|Du^{\frac{1+\gamma-2\mu}{2}}(x)| &= \frac{1+\gamma-2\mu}{2} u^{\frac{\gamma-1-2\mu}{2}}(x)|Du(x)| \\
&\leq \Theta_4 \mathrm{d}(x, \partial\Omega)^{\frac{\gamma-1-2\mu}{1+\gamma-2\mu}} \mathrm{d}(x, \partial\Omega)^{\frac{1-\gamma+2\mu}{1+\gamma-2\mu}} = \Theta_4,
\end{aligned}$$

whenever $\mathrm{d}(x, \partial\Omega)$ is small enough. In turn, since $u^{\frac{1+\gamma-2\mu}{2}}$ is Lipschitz continuous, $0 < 2/(1+\gamma-2\mu) < 1$ and $u = (u^{(1+\gamma-2\mu)/2})^{2/(1+\gamma-2\mu)}$ it follows that u is Hölder continuous up to the boundary $\partial\Omega$ with exponent $2/(1+\gamma-2\mu)$, as desired. Finally, concerning the Sobolev regularity of the solution u, observe that in light of [13, Theorem 1.3-J2], a necessary and sufficient condition for v to belong to $H^1_0(\Omega)$ is that

$$\lim_{s \to 0^+} \int_s^1 \phi(\xi) h(\phi(\xi)) d\xi < +\infty,$$

and this, since by (2.3) and (2.8) $\phi(\tau)h(\phi(\tau)) \sim \tau^{\frac{2-2\gamma}{1+\gamma-2\mu}}$ as $\tau \to 0^+$, is satisfied if and only if $\gamma < 3 - 2\mu$. In turn, $u = g(v) \in H^1_0(\Omega)$ if $\gamma < 3 - 2\mu$ since g is Lipschitz continuous on $(0, +\infty)$ being a bounded away from zero. This concludes the proof of the theorem. $\square$

Remark 2.1. We know from Theorem 1.1 that a solution u exists unique and it is Lipschitz continuous up to $\partial\Omega$ provided that $1 < \gamma \leq 1 + 2\mu$. On the other hand, if $\gamma < 1$, combining (2.3) with [2, Theorem 2.25] any solution v of $-\Delta v = h(v)$ is Lipschitz continuous, and so is u, since $u = g(v)$ and g is Lipschitz continuous on $(0, +\infty)$ since a is bounded away from zero.

Remark 2.2. Using [13, Theorem 1.3] in place of [2, Theorem 2.2 and Theorem 2.5] we could also state some global estimates for u and $|Du|$ which are valid on the whole $\bar{\Omega}$ and not only in a small neighborhood of the boundary $\partial\Omega$. Precisely, under the assumptions of Theorem 1.1, there exist $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4 > 0$ with

$$g(\Lambda_1\phi(\mathrm{d}(x, \partial\Omega))) \leq u(x) \leq g(\Lambda_2\phi(\mathrm{d}(x, \partial\Omega))), \quad \text{for } x \in \bar{\Omega},$$

$$|Du(x)| \leq \Lambda_3 \frac{\big[\mathrm{d}(x, \partial\Omega)h(\Lambda_4\phi(\mathrm{d}(x, \partial\Omega))) + \phi(\mathrm{d}(x, \partial\Omega))/\mathrm{d}(x, \partial\Omega)\big]}{\sqrt{a(u(x))}}, \quad \text{for } x \in \bar{\Omega},$$

where ϕ denotes the solution to problem (2.6). These formulas are obtained though the monotonicity of g and from $|Du(x)| = |Dv(x)|/\sqrt{a(u(x))}$ for all $x \in \bar{\Omega}$, following by the relation $u = g(v)$.

Remark 2.3. Let v be a smooth positive solution to $-\Delta v = h(v)$ in Ω, $v = 0$ on $\partial\Omega$, where $h > 0$ is as in the proof of Theorem 1.1. Let us consider the map $\psi : [0, +\infty) \to [0, +\infty)$ defined by

$$\psi(s) := \int_0^s \frac{1}{\sqrt{2\mathcal{H}(\xi)}} d\xi, \qquad \mathcal{H}(s) := \int_s^{+\infty} h(\xi) d\xi, \quad \text{for } s > 0.$$

It follows that $\psi'(s) = 1/\sqrt{2\mathcal{H}(s)} > 0$, $\psi''(s) = h(s)/(2\mathcal{H}(s))^{3/2} > 0$ and ψ satisfies $\psi''(s) = (\psi'(s))^3 h(s)$ for all $s > 0$. Put $w := -\psi(v) < 0$, we have $|Dv|^2 = |Dw|^2/(\psi'(v))^2$ as well as $\Delta w = -\psi''(v)|Dv|^2 + \psi'(v)h(v) = \psi'(v)h(v)(1 - |Dw|^2)$, yielding

$$\Delta w = f(w, Dw), \qquad f(w, Dw) := -\psi'(\psi^{-1}(-w))h(\psi^{-1}(-w))(1 - |Dw|^2),$$

with $w = 0$ on $\partial\Omega$. Whenever $f > 0$ and $z \mapsto 1/f(z, \cdot)$ is convex, one typically obtains some convexity of w if Ω is convex (see [6]) and in turn some convexity of superlevels of u since $u = g \circ \psi^{-1}(-w)$ and since $s \mapsto g \circ \psi^{-1}(s)$ is strictly increasing. See [1, Sec. 3] for the particular case $a \equiv 1$ and $f(s) = s^{-\gamma}$.

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